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Stability and mechanism order of isotropic prestressed surfaces

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Abstract

The form-finding of architectural fabric membranes may be achieved using various methods. However, this analysis always requires the calculation of stable shapes, and knowledge of the order of the possible mechanisms may provide the designer with useful data. The "surface stress density method" has been proposed as an effective form-finding tool for the design of fabric membranes. It enables tensile shapes to be determined by considering the isotropic prestress tensors in the membrane. The first objective of this paper is to demonstrate that the forms calculated in accordance with this mechanical property are stable. The second is to calculate their mechanism order. The approach is based on an energy criterion, pointed out by writing out the potential strain energy of the system and by using Lejeune–Dirichlet's theorem. This leads to defining suitable stability criteria in the vectorial subspace of the mechanisms and in the vectorial subspace orthogonal to the mechanisms. We also determine that the mechanisms in the system are of order one. The case of tensile cable nets is also analyzed.

Keywords: Stability; Mechanism order; Tensile structure; Fabric membrane; Cable net; Surface stress density method

1. Introduction

Fabric membranes and, more generally, tensile structures are characterized by their dependence on shape and prestress distribution, since an internal equilibrium must be respected initially. Several methods have been proposed to determine the two parameters, geometry and prestress, during what is called the "form-finding" process.

Several investigations have been carried out by physical modeling, such as F. Otto's work (1973) on soap films. Today, form-finding is carried out using numerical techniques.

The main methods deal with finite element procedures for large displacements (Haug and Powell, 1971) or with the dynamic relaxation approach (Barnes, 1975 and Lewis, 1996). However, one of the most common processes is the force density method (Sheck, 1974). This approach allows several equilibrium equations to be linearized but is limited to the case of tensile cable nets. Thus, Maurin and Motro (1998)

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Nomenclature

$(\vec{x}_e \vec{y}_e \vec{z}_e)$ element e local axis
$(\vec{X}\vec{Y}\vec{Z})$ system global axis
$[\ ]$, $[\ ]^t$ matrix and transpose matrix
$\{\ \}$, $<\ >$ column vector and line vector ($\{\ \}^t = <\ >$)
$\|\ \|$ vector Euclidian norm
n $n = 6$ for a cable element and $n = 9$ for a membrane element
$\{\delta^e\}$, $\{\delta\}$ elementary (n) and assembled (N) vector of node displacements (m)
$\{f^e\}$,$\{F\}$ elementary (n) and assembled (N) vector of nodal internal forces (N)
$[a^e]$,$[\mathscr{A}]$ elementary ($n \times 3$) and assembled ($N \times 3E$) equilibrium matrix of the structure (m^2)
$[k_l^e]$, $[K_l]$ elementary ($n \times n$) and assembled ($N \times N$) linear stiffness matrix ($N\,m^{-1}$)
$[d^e]$,$[\mathscr{D}]$ elementary ($n \times n$) and assembled ($N \times N$) energy characterization matrix ($N\,m^{-1}$)
$\mathrm{Ker}\mathscr{A}^t$, $\mathrm{Im}\mathscr{A}$ vectorial nullspace and column space of $[\mathscr{A}]$
$\{\delta_K\}$, $\{\delta_I\}$ assembled vector (N) of mechanisms and orthogonal to mechanisms (m)
$\{\varepsilon^e\}$,$\{\varepsilon_\ell^e\}$ elementary linear Green–Lagrange strain global (6) and local (3) vector
$\{\varepsilon_K^e\}$, $\{\varepsilon_I^e\}$ elementary global linear G–L strain vector (6) on $\mathrm{Ker}\mathscr{A}^t$ and on $\mathrm{Im}\mathscr{A}$
$\{\varepsilon_K\ell^e\}$, $\{\varepsilon_I\ell^e\}$ elementary local linear G–L strain vector (3) on $\mathrm{Ker}\mathscr{A}^t$ and on $\mathrm{Im}\mathscr{A}$
$\{\sigma_{0\ell}^e\}$, $\{\sigma_{0\ell}\}$ elementary (3) and assembled (3E) local prestress vector ($N\,m^{-2}$)
$\{\sigma_0^e\}$, $\{\sigma_0\}$ elementary (6) and assembled (6E) global prestress vector ($N\,m^{-2}$)
$\{\sigma^e\}$, $\{\sigma\}$ elementary (6) and assembled (6E) global stress vector ($N\,m^{-2}$)
$W(\delta)$ total strain energy (N m)
$W(\delta_K)$, $W(\delta_I)$ strain energy on $\mathrm{Ker}\mathscr{A}^t$ and on $\mathrm{Im}\mathscr{A}$ (N m)
$[T_\sigma^e]$, $[T_\varepsilon^e]$ elementary stress (6×3) and strain (3×6) transformation matrix
$[b_l^e]$ elementary linear displacements interpolation ($6 \times n$) matrix (m^{-1})
$[E^e]$ elementary material elastic coefficients matrix (6×6) matrix ($N\,m^{-2}$)
N, E number of degrees of freedom (d.o.f.) and number of elements
s_e cable cross-section area or membrane surface area (m^2)
t_e membrane thickness (m)
v_e element volume (m^3)
L_e, L'_e cable initial length and after displacement length (m)
L_{ei} membrane side i initial length (m)
Δ^e, Δ_i^e cable length variation and membrane side i length variation (m)
ε_i^e membrane side i strain

have developed the "surface stress density method" whereby the force density method is generalized to the case of surfaces. It is characterized by the use of isotropic prestress tensors in the membrane.

Nevertheless, the stability issue of the determined shapes is not systematically studied by the authors. This remains highly important since a useful and reliable shape-finding method requires that it determine only stable forms. Moreover, tensile systems may have at least one mechanism and it is also important to know the order of this mechanism. This value indeed represents the maximum deformation of the elements in the system when a displacement follows the mechanism: a high order (infinite at the limit) implies low deformation (no deformation at the limit i.e., a solid body movement). Hence, knowing the order may help the designer to evaluate the mechanical behavior of the system if such a situation arises.

The first purpose of this paper is thus to demonstrate that a membrane surface determined by using isotropic prestress tensors is always stable. The next step will be to calculate the order of the possible mechanisms.

The paper hence deals with the case of isotropic membranes as well as that of tensile cable nets. After some considerations on equilibrium stability, we will define the mechanical modeling used for these systems and the mathematical formulations used to determine the mechanisms.

The approach will lead us to writing out the "elementary energy characterization matrix" followed by the necessary stability criteria. We will subsequently verify whether tensile isotropic surfaces and cable nets satisfy these criteria. This will also enable us to determine the order of the mechanism according to energy characterization (Vassart et al., 2000).

2. Equilibrium stability

Lejeune–Dirichlet's theorem can be used to ascertain the stability of a conservative system. It demonstrates that an equilibrium position is stable if its potential energy is strictly minimal. If we consider the particular case of an unloaded structure (no external loading such as wind or snow), then the potential energy corresponds to the internal strain energy W.

The geometry of a prestressed structure is determined by using a form-finding method that allows the calculation of its shape and prestress. This provides the "reference configuration" that is characterized by a nil strain energy ($W = 0$ because no displacement occurs). Hence, when a compatible virtual displacement δ is considered in the vicinity of this reference geometry, the strain energy is of a strict minimum only and only if the increment $W(\delta)$ is positive definite.

Moreover, Liapounov (Knops and Wilkes, 1973) has demonstrated that the positive definitiveness of an analytic function depends on the terms of its lowest degree of development. This means that if the main part of the function $W(\delta)$ is positive definite, then $W(\delta)$ is also positive definite. As a consequence, only the main part of strain energy will be considered in our calculations.

The assumption that virtual displacement relates to the vicinity of the reference geometry implies that the study respects the small displacements hypothesis. The displacement δ is thus of an order inferior or equal to one, that is to say $\|\delta\| \leqslant \mathrm{O}_1$ by using the vector Euclidian norm. More generally, we will write the order r as $\mathrm{O}_r = \xi \mathrm{O}_{r-1} = \xi^r \mathrm{O}_0$. The order O_0 corresponds to the order zero (matching up to the lengths of the elements in the system) and ξ is a strictly positive real number that is very small with respect to one ($\xi \ll 1$).

The written form $\overset{r}{\approx}$ will be used for an equality limited to order r in its development; the symbol $\approx$ will be considered as equivalent to $\overset{1}{\approx}$.

As a conclusion, the stability criterion for a structure requires that $\forall \delta \in (\Re^N - 0)$ with $\|\delta\| \leqslant \mathrm{O}_1$ so $W(\delta) > 0$ must be verified.

3. Mechanical modeling of isotropic prestressed systems

3.1. Representation of structural prestress

Through discrete modeling using finite elements it is possible to characterize the prestress state of a structure by considering the Cauchy prestress tensors for each element. These tensors are usually written in the local axis ($\vec{x}_e$ $\vec{y}_e$ $\vec{z}_e$) associated to the elements and according to the column vector $\{\sigma^e_{0\ell}\}$ (the notation ℓ shows a "local axis" writing).

For a cable element, prestress has only one non-nil component along the $\vec{x}_e$ axis.

Thus, a tensile cable element is written as $\{\sigma^e_{0\ell}\}^t = \langle\, \sigma_{0x_e} \quad \sigma_{0y_e} \quad \sigma_{0xy_e} \,\rangle = \langle\, \sigma_{0x_e} \quad 0 \quad 0 \,\rangle$ so that $\{\sigma^e_{0\ell}\}^t = \langle\, \sigma^e_{0\ell} \quad 0 \quad 0 \,\rangle$ with $\sigma^e_{0\ell} > 0$.

In the case of tensile membranes, the prestress for an element is characterized in its local axis by $\{\sigma^e_{0\ell}\}^t = \langle\, \sigma_{0x_e} \quad \sigma_{0y_e} \quad \sigma_{0xy_e} \,\rangle$. However, if we consider the particular situation of an isotropic prestressed membrane, the shear prestress σ_{0xy_e} will be equal to zero and $\sigma_{0x_e} = \sigma_{0y_e}$. For a membrane element, this implies that $\{\sigma^e_{0\ell}\}^t = \langle\, \sigma^e_{0\ell} \quad \sigma^e_{0\ell} \quad 0 \,\rangle$ with $\sigma^e_{0\ell} > 0$.

3.2. Search for mechanisms

When a structure is discretized with finite elements, the internal pretension forces acting on the nodes of an element may be calculated according to:

$$\{f^e\} = v_e[b^e_l]^t\{\sigma^e_0\} = v_e[b^e_l]^t[T^e_\sigma]\{\sigma^e_{0\ell}\} = [a^e]\{\sigma^e_{0\ell}\} \tag{1}$$

where v_e represents the volume of the element, $[b^e_l]$ the linear displacement interpolation matrix and $[T^e_\sigma]$ the "local to global" stress transformation matrix.

All the vectors and matrices are written in the global axis of the structure ($\vec{X}\vec{Y}\vec{Z}$), except for quantities using the suffix ℓ which are written in the local axis of the element.

By assembling the elementary equation (1), we get:

$$\{F\} = [\mathscr{A}]\{\sigma_{0\ell}\} = \{0\} \tag{2}$$

This relationship shows that the reference position is in equilibrium and $\{\sigma_{0\ell}\}$ characterizes the selfstress vector of the entire system (written by considering the elementary values $\sigma^e_{0\ell}$). The matrix $[\mathscr{A}]$ is called the "equilibrium matrix" of the structure.

If we consider a compatible virtual displacement $\delta = \{\delta\}$ of the nodes (vectorial writing), the linearized strain tensor of Green–Lagrange may be determined by:

$$\{\varepsilon^e_\ell\} = [T^e_\varepsilon]\{\varepsilon^e\} \approx [T^e_\varepsilon][b^e_l]\{\delta^e\} \tag{3}$$

where $[T^e_\varepsilon] = [T^e_\sigma]^t$ corresponds to the "global to local" strain transformation matrix. Thus:

$$v_e\{\varepsilon^e_\ell\} \approx v_e[T^e_\sigma]^t[b^e_l]\{\delta^e\} = [a^e]^t\{\delta^e\} \tag{4}$$

We then assemble the elementary relationships (4) with $\{\varepsilon^{v_e}_\ell\} = v_e\{\varepsilon^e_\ell\}$ giving:

$$\{\varepsilon^v_\ell\} \approx [\mathscr{A}]^t\{\delta\} \tag{5}$$

The kernel of $[\mathscr{A}]^t$, written $\mathrm{Ker}\mathscr{A}^t$, defines the vectorial subspace of the mechanisms. It corresponds to the vectors $\{\delta\}$ that verify $\{\varepsilon^{v_e}_\ell\} = \{0\}$.

However, it can be demonstrated (Vassart et al., 2000) that, in the displacement space $\Re^N$, the vectorial subspace $\mathrm{Im}\mathscr{A}$ (the column space of matrix $[\mathscr{A}]$) is orthogonal and supplementary to the mechanisms. Hence, we have $\Re^N = \mathrm{Ker}\mathscr{A}^t \oplus \mathrm{Im}\mathscr{A}$ where $\oplus$ is the direct summation. Each displacement can thus be uniquely split up into:

$$\{\delta\} = \{\delta_K\} + \{\delta_I\} \text{ with } \{\delta_K\} \in \mathrm{Ker}\mathscr{A}^t \text{ and } \{\delta_I\} \in \mathrm{Im}\mathscr{A} \tag{6}$$

4. Determining elementary strain energy in the subspace of mechanisms

We will now consider the case where the displacement corresponds with a mechanism and where no orthogonal displacement $\{\delta_I\}$ occurs. This means that $\{\delta\} = \{\delta_K\}$ of order $r \geqslant 1$.

The objective is to propose a matrix writing of the elementary strain energy $W(\delta_K^e)$ where the virtual displacement corresponds with the mechanism $\{\delta_K^e\}$.

4.1. Case of prestressed cable element

The displacement vector of nodes 1 and 2 for a cable element e of initial length L_e is:

$$\{\delta_K^e\}^t = \langle\vec{\delta}_1|\vec{\delta}_2\rangle = \langle\delta_{1_X}\delta_{1_Y}\delta_{1_Z}|\delta_{2_X}\delta_{2_Y}\delta_{2_Z}\rangle$$

(see Fig. 1). After deformation the member length (with $\vec{\delta}_{12} = \vec{\delta}_1 - \vec{\delta}_2$) becomes:

$$L_e'^2 = \overrightarrow{1'2'}^2 = (X_{12} + \delta_{12_X})^2 + (Y_{12} + \delta_{12_Y})^2 + (Z_{12} + \delta_{12_Z})^2 = L_e^2 + \vec{\delta}_{12}^2 + 2\vec{\delta}_{12}\cdot\overrightarrow{12} \tag{7}$$

Since $\vec{\delta}_{12}\cdot\vec{x}_e = 0$ by definition of the vectorial subspace Ker$\mathscr{A}^t$, it follows that:

$$L_e' = L_e\left(1 + \frac{\vec{\delta}_{12}^2}{L_e^2}\right)^{1/2} \overset{2r}{\approx} L_e + \frac{\vec{\delta}_{12}^2}{2L_e} \tag{8}$$

Cable length variation is thus:

$$\Delta^e = L_e' - L_e \overset{2r}{\approx} \frac{\vec{\delta}_{12}^2}{2L_e} \tag{9}$$

And cable strain, written in its local axis:

$$\{\varepsilon_{K\ell}^e\}^t = <\ \varepsilon_{Kx_e}\quad \varepsilon_{Ky_e}\quad \varepsilon_{Kxy_e}\ > = <\ \varepsilon_{Kx_e}\quad 0\quad 0\ > \text{ with } \varepsilon_{Kx_e} \overset{2r}{\approx} \frac{\vec{\delta}_{12}^2}{2L_e^2} \tag{10}$$

The elementary strain energy of a prestressed cable in the mechanism subspace is:

$$W(\delta_K^e) = v_e\sigma_{0x_e}\varepsilon_{Kx_e} \tag{11}$$

where $v_e = s_eL_e$ is the cable volume and s_e its cross-section area.

We can then write the following matrix form:

$$W(\delta_K^e) \overset{2r}{\approx} \frac{1}{2}\{\delta_K^e\}^t[d^e]\{\delta_K^e\} \tag{12}$$

with

$$[d^e] = \frac{v_e\sigma_{0\ell}^e}{L_e^2}\begin{bmatrix}[\mathrm{Id}_3] & -[\mathrm{Id}_3]\\ -[\mathrm{Id}_3] & [\mathrm{Id}_3]\end{bmatrix} \text{ where } [\mathrm{Id}_3] \text{ is the } (3\times 3) \text{ identity matrix} \tag{13}$$

We propose calling the symmetric matrix $[d^e]$ the "elementary energy characterization matrix".

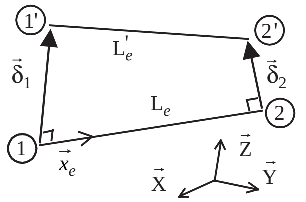

Fig. 1. Deformation of a cable element in Ker$\mathscr{A}^t$.

4.2. Case of isotropic prestressed membrane element

The displacement vector for nodes 1, 2 and 3 of a membrane element e with an initial area s_e is (see Fig. 2):

$$\{\delta_K^e\}^t = \langle\vec{\delta}_1|\vec{\delta}_2|\vec{\delta}_3\rangle = \langle\delta_{1_X}\delta_{1_Y}\delta_{1_Z}|\delta_{2_X}\delta_{2_Y}\delta_{2_Z}|\delta_{3_X}\delta_{3_Y}\delta_{3_Z}\rangle$$

The length variations of the three sides are:

$$\Delta_1^e \overset{2r}{\approx} \frac{\vec{\delta}_{12}^2}{2L_{e1}}, \quad \Delta_2^e \overset{2r}{\approx} \frac{\vec{\delta}_{23}^2}{2L_{e2}}, \quad \Delta_3^e \overset{2r}{\approx} \frac{\vec{\delta}_{31}^2}{2L_{e3}} \tag{14}$$

And the associated strains of the sides are:

$$\varepsilon_i^e = \frac{\Delta_i^e}{L_{ei}} \tag{15}$$

The membrane strain may thus be written, in its local axis, as:

$$\{\varepsilon_{K\ell}^e\} = \begin{Bmatrix} \varepsilon_{Kx_e} \\ \varepsilon_{Ky_e} \\ \varepsilon_{Kxy_e} \end{Bmatrix} \overset{2r}{\approx} \frac{1}{b_2c_3 - c_2b_3} \begin{bmatrix} b_2c_3 - c_2b_3 & 0 & 0 \\ a_3c_2 - a_2c_3 & c_3 & -c_2 \\ a_2b_3 - a_3b_2 & -b_3 & b_2 \end{bmatrix} \begin{Bmatrix} \varepsilon_1^e \\ \varepsilon_2^e \\ \varepsilon_3^e \end{Bmatrix} \tag{16}$$

with the coefficients:

$$a_i = \cos^2\theta_i, \quad b_i = \sin^2\theta_i, \quad c_i = \cos\theta_i\sin\theta_i \tag{17}$$

where θ_i is the oriented angle between the local axis $\vec{x}_e$ and the side i.

Moreover, with $\psi = \theta_2 - \theta_3$ we have:

$$b_2c_3 - c_2b_3 = \sin\theta_2\sin\theta_3\sin\psi > 0 \tag{18}$$

This leads to:

$$\begin{Bmatrix} \varepsilon_{Kx_e} \\ \varepsilon_{Ky_e} \end{Bmatrix} \overset{2r}{\approx} \frac{1}{2} \begin{bmatrix} \frac{1}{L_{e1}^2} & 0 & 0 \\ \frac{1}{L_{e1}^2 tg\theta_2 tg\theta_3} & \frac{1}{2s_e tg\theta_3} & \frac{-1}{2s_e tg\theta_2} \end{bmatrix} \begin{Bmatrix} \vec{\delta}_{12}^2 \\ \vec{\delta}_{23}^2 \\ \vec{\delta}_{31}^2 \end{Bmatrix} = \frac{1}{2}[m^e] \begin{Bmatrix} \vec{\delta}_{12}^2 \\ \vec{\delta}_{23}^2 \\ \vec{\delta}_{31}^2 \end{Bmatrix} \tag{19}$$

We chose to write the matrix $[m^e]$ as:

$$[m^e] = \begin{bmatrix} m_{11}^e & 0 & 0 \\ m_{21}^e & m_2^e & m_3^e \end{bmatrix} \tag{20}$$

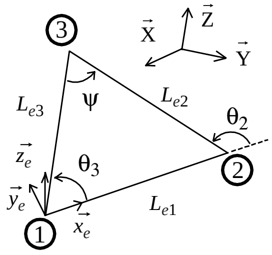

Fig. 2. Membrane element geometry.

By definition, the strain energy on the vectorial subspace of mechanisms $\mathrm{Ker}\mathscr{A}^t$ for a prestressed membrane element is:

$$W(\delta_{\mathrm{K}}^{\mathrm{e}}) = v_{\mathrm{e}}\{\sigma_{0\ell}^{\mathrm{e}}\}^t\{\varepsilon_{\mathrm{K}\ell}^{\mathrm{e}}\} \tag{21}$$

where $v_{\mathrm{e}} = s_{\mathrm{e}}t_{\mathrm{e}}$ corresponds to the volume of the element and t_{e} its thickness.

If we assume that the prestress on the element is isotropic, then $\{\sigma_{0\ell}^{\mathrm{e}}\}^t = \langle\, \sigma_{0\ell}^{\mathrm{e}} \quad \sigma_{0\ell}^{\mathrm{e}} \quad 0\,\rangle$ and the strain energy becomes:

$$W(\delta_{\mathrm{K}}^{\mathrm{e}}) = v_{\mathrm{e}}\sigma_{0\ell}^{\mathrm{e}}(\varepsilon_{\mathrm{K}x_{\mathrm{e}}} + \varepsilon_{\mathrm{K}y_{\mathrm{e}}}) \overset{2r}{\approx} \frac{1}{2}v_{\mathrm{e}}\sigma_{0\ell}^{\mathrm{e}}\left(m_1^{\mathrm{e}}\vec{\delta}_{12}^2 + m_2^{\mathrm{e}}\vec{\delta}_{23}^2 + m_3^{\mathrm{e}}\vec{\delta}_{31}^2\right) \tag{22}$$

with $m_1^{\mathrm{e}} = m_{11}^{\mathrm{e}} + m_{21}^{\mathrm{e}} = \frac{1}{2s_{\mathrm{e}}tg\psi}$. Moreover, the coefficients m_i^{e} verify:

$$m_1^{\mathrm{e}} + m_2^{\mathrm{e}} = \left(\frac{L_{\mathrm{e}3}}{2s_{\mathrm{e}}}\right)^2 > 0, \quad m_1^{\mathrm{e}} + m_3^{\mathrm{e}} = \left(\frac{L_{\mathrm{e}2}}{2s_{\mathrm{e}}}\right)^2 > 0, \quad m_2^{\mathrm{e}} + m_3^{\mathrm{e}} = \left(\frac{L_{\mathrm{e}1}}{2s_{\mathrm{e}}}\right)^2 > 0 \tag{23}$$

The strain energy on $\mathrm{Ker}\mathscr{A}^t$, for an isotropic prestressed membrane element, may therefore be written in the same form of Eq. (12).

We can indeed get $W(\delta_{\mathrm{K}}^{\mathrm{e}}) \overset{2r}{\approx} \frac{1}{2}\{\delta_{\mathrm{K}}^{\mathrm{e}}\}^T[d^{\mathrm{e}}]\{\delta_{\mathrm{K}}^{\mathrm{e}}\}$ where $[d^{\mathrm{e}}]$ represents the elementary energy characterization matrix:

$$[d^{\mathrm{e}}] = \frac{1}{2}v_{\mathrm{e}}\sigma_{0\ell}^{\mathrm{e}}\begin{bmatrix} (m_1^{\mathrm{e}} + m_3^{\mathrm{e}})[\mathrm{Id}_3] & -m_1^{\mathrm{e}}[\mathrm{Id}_3] & -m_3^{\mathrm{e}}[\mathrm{Id}_3] \\ -m_1^{\mathrm{e}}[\mathrm{Id}_3] & (m_1^{\mathrm{e}} + m_2^{\mathrm{e}})[\mathrm{Id}_3] & -m_2^{\mathrm{e}}[\mathrm{Id}_3] \\ -m_3^{\mathrm{e}}[\mathrm{Id}_3] & -m_2^{\mathrm{e}}[\mathrm{Id}_3] & (m_2^{\mathrm{e}} + m_3^{\mathrm{e}})[\mathrm{Id}_3] \end{bmatrix} \tag{24}$$

5. Stability analysis in the different subspaces

This section deals simultaneously with the two types of studied systems (a prestressed cable net and an isotropic prestressed membrane).

The purpose of the approach is to define a stability criterion, firstly, in the vectorial subspace orthogonal to the mechanisms $\mathrm{Im}\mathscr{A}$ and secondly, in the vectorial subspace of the mechanisms $\mathrm{Ker}\mathscr{A}^t$. We will then consider the case where the virtual displacement relates to both of these vectorial subspaces.

5.1. Stability in $\mathrm{Im}\mathscr{A}$

According to the splitting $\{\delta\} = \{\delta_{\mathrm{K}}\} + \{\delta_{\mathrm{I}}\}$, we study the situation where $\{\delta_{\mathrm{K}}\} = \{0\}$ and $\{\delta\} = \{\delta_{\mathrm{I}}\}$ of order $r \geqslant 1$. The elementary strain energy is:

$$W(\delta_{\mathrm{I}}^{\mathrm{e}}) = v_{\mathrm{e}}\{\sigma_0^{\mathrm{e}}\}^t\{\varepsilon_{\mathrm{I}}^{\mathrm{e}}\} + \frac{1}{2}v_{\mathrm{e}}\{\sigma^{\mathrm{e}}\}^t\{\varepsilon_{\mathrm{I}}^{\mathrm{e}}\} \tag{25}$$

We assume that the behavior of the material is elastic linear; the matrix $[E^{\mathrm{e}}]$ represents the material elastic coefficients. Moreover, we also assume that these coefficients are of the order $\mathrm{O}_{-1} = \mathrm{O}_0/\xi$ (Vassart et al., 2000). It follows that:

$$\{\varepsilon_{\mathrm{I}}^{\mathrm{e}}\} \overset{r}{\approx} [b_l^{\mathrm{e}}]\{\delta_{\mathrm{I}}^{\mathrm{e}}\} \text{ and } \{\sigma^{\mathrm{e}}\} = [E^{\mathrm{e}}]\{\varepsilon_{\mathrm{I}}^{\mathrm{e}}\} \overset{r-1}{\approx} [E^{\mathrm{e}}][b_l^{\mathrm{e}}]\{\delta_{\mathrm{I}}^{\mathrm{e}}\} \tag{26}$$

The elementary deformation energy thus becomes:

$$W(\delta_{\mathrm{I}}^{\mathrm{e}}) \overset{2r-1}{\approx} v_{\mathrm{e}}\{\sigma_0^{\mathrm{e}}\}^t[b_l^{\mathrm{e}}]\{\delta_{\mathrm{I}}^{\mathrm{e}}\} + \frac{1}{2}v_{\mathrm{e}}\{\delta_{\mathrm{I}}^{\mathrm{e}}\}^t[b_l^{\mathrm{e}}]^t[E^{\mathrm{e}}][b_l^{\mathrm{e}}]\{\delta_{\mathrm{I}}^{\mathrm{e}}\} \tag{27}$$

If $[k_l^e]$ represents the elementary linear stiffness matrix, we can write:

$$W(\delta_I^e) \overset{2r-1}{\approx} v_e\{\sigma_{0\ell}^e\}^t[T_\sigma^e]^t[b_l^e]\{\delta_I^e\} + \frac{1}{2}\{\delta_I^e\}^t[k_l^e]\{\delta_I^e\} \tag{28}$$

$$W(\delta_I^e) \overset{2r-1}{\approx} \{\sigma_{0\ell}^e\}^t[a^e]^t\{\delta_I^e\} + \frac{1}{2}\{\delta_I^e\}^t[k_l^e]\{\delta_I^e\} \tag{29}$$

Assembling the relationships (29) gives:

$$W(\delta_I) \overset{2r-1}{\approx} \{\sigma_{0\ell}\}^t[\mathscr{A}]^t\{\delta_I\} + \frac{1}{2}\{\delta_I\}^t[K_l]\{\delta_I\} \tag{30}$$

Since Eq. (2) shows the structure equilibrium by $[\mathscr{A}]\{\sigma_{0\ell}\} = \{0\}$, the strain energy of the global system is therefore:

$$W(\delta_I) \overset{2r-1}{\approx} \frac{1}{2}\{\delta_I\}^t[K_l]\{\delta_I\} \tag{31}$$

If suitable boundary conditions have been specified to the structure (i.e., no rigid body movement), then the global linear stiffness matrix $[K_l]$ is positive definite.

We can deduce, in such a case, that the strain energy $W(\delta_I)$ is also positive definite and thus conclude that a tensile system is always stable when virtual displacement does not relate to the vectorial subspace of the mechanisms.

5.2. Stability in Ker$\mathscr{A}^t$

In this case, we consider that $\{\delta\} = \{\delta_K\}$ of order $r \geqslant 1$. The elementary strain energy is:

$$W(\delta_K^e) = v_e\{\sigma_0^e\}^t\{\varepsilon_K^e\} + \frac{1}{2}v_e\{\sigma^e\}^t\{\varepsilon_K^e\} \tag{32}$$

Since $\{\sigma^e\} = [E^e]\{\varepsilon_K^e\}$ and since the components of $\{\varepsilon_K^e\}$ are related to $\vec{\delta}_{ij}^2$ terms of order 2, then the multiplication $\{\sigma^e\}^t\{\varepsilon_K^e\}$ is of order $4r-1$.

By limiting the expression (32) to the main order $2r$, we obtain:

$$W(\delta_K^e) \overset{2r}{\approx} v_e\{\sigma_0^e\}^t\{\varepsilon_K^e\} = v_e\{\sigma_{0\ell}^e\}^t\{\varepsilon_{K\ell}^e\} = \frac{1}{2}\{\delta_K^e\}^t[d^e]\{\delta_K^e\} \tag{33}$$

Therefore, by assembling the elementary equation (33):

$$W\{\delta_K\} \overset{2r}{\approx} \frac{1}{2}\{\delta_K\}^t[\mathscr{D}]\{\delta_K\} \tag{34}$$

The global energy characterization matrix $[\mathscr{D}]$ is written by assembling the matrices $[d^e]$. We stress the fact that, if $[\mathscr{D}]$ is positive definite, then $W(\delta_K)$ is also positive definite.

Moreover, if a displacement $\delta_K \neq 0$ verifies at the second order that $\{\delta_K\}^t[\mathscr{D}]\{\delta_K\}^2 \neq 0$, it is not necessary to take into account the terms of δ_K of an order superior to one so as to calculate $W(\delta_K)$. This means that, in such a case, the system has mechanisms of order one only.

The stability criterion in Ker$\mathscr{A}^t$ may be stated thus:

> "An isotropic tensile system with a virtual displacement in the vectorial subspace of its mechanisms is stable if, and only if, its energy characterization matrix is positive definite. If such a condition is verified, the mechanisms are of order one".

The study of the positive definitiveness of the matrix $[\mathscr{D}]$ will be carried out in Section 5.5.

After the definition of the stability criteria in the two vectorial subspaces Im$\mathscr{A}$ and Ker$\mathscr{A}^t$, we propose carrying out a study on their vicinity. Generally speaking, a virtual displacement does indeed relate to both of these subspaces.

5.3. Stability in the vicinity of Im$\mathscr{A}$

We consider $\{\delta\} = \{\delta_K\} + \{\delta_I\}$ with $\{\delta_I\}$ of order one and $\{\delta_K\}$ of order $r \geqslant 1$. Since $\{\varepsilon^e\} = \{\varepsilon^e_K\} + \{\varepsilon^e_I\}$ the elementary strain energy is:

$$W(\delta^e) = v_e\{\sigma^e_0\}^t\{\varepsilon^e\} + \frac{1}{2}v_e\{\sigma^e\}^t\{\varepsilon^e\} \tag{35}$$

By limiting this relationship to the main order, we have:

$$W(\delta^e) \overset{2r}{\approx} \frac{1}{2}\{\delta^e_I\}^t[k^e_l]\{\delta^e_I\} + \frac{1}{2}\{\delta^e_K\}^t[d^e]\{\delta^e_K\} \tag{36}$$

and, by assembling the relations:

$$W(\delta) \overset{2r}{\approx} \frac{1}{2}\underbrace{\{\delta_I\}^t[K_l]\{\delta_I\}}_{\text{order 1}} + \frac{1}{2}\underbrace{\{\delta_K\}^t[\mathscr{D}]\{\delta_K\}}_{\text{order }2r} \tag{37}$$

We note that $(W\delta)$ is always positive definite because the matrix $[K_l]$ is positive definite.

Nevertheless, this does not automatically imply the stability of the system. The vicinity of Im$\mathscr{A}$ could indeed correspond to the vicinity of a mechanism. The stability will then be dependent on this other Ker$\mathscr{A}^t$ vicinity.

5.4. Stability in the vicinity of Ker$\mathscr{A}^t$

In this case $\{\delta\} = \{\delta_I\} + \{\delta_K\}$ with $\{\delta_I\}$ of order $r \geqslant 1$ and $\{\delta_K\}$ of order one.

According to a similar approach and by limiting the formula to the main order, we obtain:

$$W(\delta) \overset{\alpha}{\approx} \frac{1}{2}\underbrace{\{\delta_I\}^t[K_l]\{\delta_I\}}_{\text{order }2r-1} + \frac{1}{2}\underbrace{\{\delta_K\}^t[\mathscr{D}]\{\delta_K\}}_{\text{order 2}} \quad \text{with } \alpha = \max(2r-1; 2) \tag{38}$$

We point up that whatever order r of $\{\delta_I\}$ is chosen, $W(\delta)$ is always positive definite if the matrix $[\mathscr{D}]$ is positive definite.

The stability criterion for an isotropic tensile system could therefore be linked to the analysis of positive definitiveness of the energy characterization matrix in the vectorial subspace of the mechanisms Ker$\mathscr{A}^t$.

5.5. Positive definitiveness study of the energy characterization matrix

5.5.1. Case of prestressed cable net

The matrix $[\mathscr{D}]$ is positive definite in Ker$\mathscr{A}^t$ if $\forall\delta \in (\text{Ker}\mathscr{A}^t - 0)$ of order r so that $W(\delta) \overset{2r}{\approx} \{\delta\}^t \times [\mathscr{D}]\{\delta\} > 0$.

We see that $\mathscr{D}_{ij} = \mathscr{D}_{ji} < 0$ since $[d^e]$ matrices are symmetric. Moreover, the elimination of the lines and columns of $[\mathscr{D}]$ by assembling these elementary matrices according to suitable boundary conditions (likewise the matrix $[K_l]$ assembly), leads to $\mathscr{D}_{ii} \geqslant \sum_{j=1(j\neq i)}^{N} - \mathscr{D}_{ij} > 0$.

This may be written as the relation $\mathscr{D}_{ii} = \sum_{j=1(j\neq i)}^{N} - \mathscr{D}_{ij} + \mathscr{D}_{ii}^{*}$ with $\mathscr{D}_{ii}^{*} > 0$. Thus, we have:

$$W(\delta) \overset{2r}{\approx} \sum_{i=1}^{N}\left(\delta_i^2 \mathscr{D}_{ii} + \delta_i \sum_{j=1(j\neq i)}^{N} \delta_j \mathscr{D}_{ij}\right) = \sum_{i=1}^{N}\left(-\delta_i^2 \sum_{j=1(j\neq i)}^{N} \mathscr{D}_{ij} + \delta_i \sum_{j=1(j\neq i)}^{N} \delta_j \mathscr{D}_{ij} + \delta_i^2 \mathscr{D}_{ii}^{*}\right) \tag{39}$$

since

$$\sum_{i=1}^{N}\left(\delta_i^2 \sum_{j=1(j\neq i)}^{N} \mathscr{D}_{ij}\right) = \sum_{i=1}^{N}\left(\sum_{j=i+1}^{N} (\delta_i^2 + \delta_j^2) \mathscr{D}_{ij}\right) \tag{40}$$

and

$$\sum_{i=1}^{N} \delta_i \left(\sum_{j=1(j\neq i)}^{N} \delta_j \mathscr{D}_{ij}\right) = 2\sum_{i=1}^{N}\left(\sum_{j=i+1}^{N} \delta_i \delta_j \mathscr{D}_{ij}\right) \tag{41}$$

the strain energy may be written as:

$$W(\delta) \overset{2r}{\approx} \sum_{i=1}^{N}\left(\sum_{j=i+1}^{N} \mathscr{D}_{ij}(-\delta_i^2 + 2\delta_i\delta_j - \delta_j^2) + \delta_i^2 \mathscr{D}_{ii}^{*}\right) \tag{42}$$

This leads to:

$$W(\delta) \overset{2r}{\approx} \sum_{i=1}^{N}\left(-\sum_{j=i+1}^{N} \mathscr{D}_{ij}\delta_{ij}^2 + \delta_i^2 \mathscr{D}_{ii}^{*}\right) > 0 \tag{43}$$

The energy characterization matrix positive is therefore positive definite.

This analysis permits the verification that $\{\delta_{\mathrm{K}}\}^{t}[\mathscr{D}]\{\delta_{\mathrm{K}}\}^{2} \neq 0$ for a displacement $\delta_{\mathrm{K}} \neq 0$ of order one and then concludes that the mechanisms of a tensile cable net are always of order one.

We may however note that, if the boundary conditions are not suitable or are not taken into account during the assembly, the matrix $[\mathscr{D}]$ is only semi positive definite. This case signifies that there is a degree of freedom i with $\mathscr{D}_{ii}^{*} = 0$ and another degree of freedom j such that, for $\delta_i = \delta_j$, we have $W(\delta) = 0$ (i.e., a rigid body movement).

5.5.2. Case of isotropic prestressed membrane element

Although the approach is slightly different here, the above remark for cable net may be applied. An analysis of displacements for a membrane element results in:

$$\vec{\delta}_{12}^2 = \vec{\delta}_{23}^2 + \vec{\delta}_{31}^2 - 2\|\vec{\delta}_{23}\|\|\vec{\delta}_{31}\|, \quad \vec{\delta}_{23}^2 = \vec{\delta}_{12}^2 + \vec{\delta}_{31}^2 - 2\|\vec{\delta}_{12}\|\|\vec{\delta}_{31}\|$$

and

$$\vec{\delta}_{31}^2 = \vec{\delta}_{12}^2 + \vec{\delta}_{23}^2 - 2\|\vec{\delta}_{12}\|\|\vec{\delta}_{23}\| \tag{44}$$

Study is based on the matrix values $[m^{\mathrm{e}}]$ defined in (20) and the relations (22).

If $\forall m_2^{\mathrm{e}}$ we verify $m_1^{\mathrm{e}} \geqslant 0$ and $m_3^{\mathrm{e}} \leqslant 0$, so that the elementary strain energy may be written as:

$$W(\delta_{\mathrm{K}}^{\mathrm{e}}) \overset{2r}{\approx} \frac{1}{2} v_{\mathrm{e}} \sigma_{0\ell}^{\mathrm{e}} \left(\vec{\delta}_{23}^2 (m_1^{\mathrm{e}} + m_3^{\mathrm{e}}) + \vec{\delta}_{31}^2 (m_2^{\mathrm{e}} + m_3^{\mathrm{e}}) - 2m_3^{\mathrm{e}} \|\vec{\delta}_{23}\|\|\vec{\delta}_{31}\|\right) \tag{45}$$

According to the relationships (23), we therefore obtain $W(\delta_{\mathrm{K}}^{\mathrm{e}}) \geqslant 0$.

Likewise, if $m_1^e \leqslant 0$ and $m_3^e \leqslant 0$ then:

$$W(\delta_K^e) \overset{2r}{\approx} \frac{1}{2} v_e \sigma_{0\ell}^e \left(\vec{\delta}_{12}^2 \left(m_1^e + m_3^e \right) + \vec{\delta}_{31}^2 \left(m_1^e + m_2^e \right) - 2 m_1^e \|\vec{\delta}_{12}\| \|\vec{\delta}_{31}\| \right) \geqslant 0 \tag{46}$$

In every situation, the strain energy verifies that $W(\delta_K^e) \geqslant 0$ and consequently, the elementary matrices $[d^e]$ are semi positive definite.

In the light of the previous remark pointed up for cable net, the choice suitable boundary conditions in assembling the matrices $[d^e]$ will lead to the positive definitiveness of the energy characterization matrix $[\mathscr{D}]$.

6. Conclusion

The design of tensile structures, fabric membranes and cable nets, could be carried out by considering isotropic prestressed systems. However, the shapes so calculated have to be stable. This paper demonstrates that a system determined according to this prestress distribution is stable.

Lejeune–Dirichlet's theorem indeed shows that, for such structures, the stability criteria are dependent on the behavior analysis within the vicinity of the mechanism subspace. By defining and writing out the energy characterization matrix, the stability criteria can be associated with the positive definitiveness of this matrix. As such a requirement is verified for isotropic prestressed structures, this demonstrates their stability and that the order of the mechanisms is equal to one.

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